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Bifactorizable wavefunctions

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Abstract. Physical motivation is given for studying properties of bifactorizable (BF) functions, i.e. functions of two variables which can be factored in two different ways. The functional equation which a BF function must satisfy is derived and the form of its solution is shown to be a Gaussian. This also yields the functional equation defining a Gaussian, in analogy to the equation $E(x+y) = E(x)E(y)$ defining the exponential function. Further, the following theorem is proved: if two systems are prepared independently, and their centre of mass is found to be in a pure state, then both systems were prepared in pure states, each of which is a Gaussian in the coordinate representation, and so are the centre of mass and relative coordinate states.

1. Introduction

Consider a classical system which is made up of two independent subsystems. The coordinate of the subsystems are labelled by $x_i = (Q_i, P_i)$, $i = 1, 2$. The system can be described by a product (factorized form) of the respective distribution functions $f_i(x_i)$:

$$f(x_1, x_2) = f_1(x_1)f_2(x_2).$$

Now suppose that the system may also be described in a factorized form in terms of other variables, x'_1, x'_2 , which are related to x_1, x_2 by a linear transformation, i.e. we also have

$$f(x_1, x_2) = f_3(x'_1)f_4(x'_2).$$

We say that $f(x_1, x_2)$ possesses the bifactorization property. An example for such a case could be when we can also factorize the *distribution function of our system* (i.e. $f(x_1, x_2)$) in terms of the centre of mass and relative coordinates. Suppose that in its factorized form one of the above distribution functions, say $f_3(x'_1)$, where x'_1 corresponds to the centre of mass variables, describes a classically 'pure state', i.e.

$$f_3(Q, P) = \delta(P - P_0)\delta(Q - Q_0)$$

which states that the position of the centre of mass Q is Q_0 , while the momentum of the centre of mass P is P_0 . Now if the total system bifactorizes, then it obviously follows that all other distribution functions are also 'pure classical states'. x_1 and x_2 being *independent* variables of product functions, fixing of their sum by a δ -function implies that each is fixed by a δ -function.

A closely related quantal problem was discussed more than 20 years ago by Aharonov *et al* [1], and more recently by Emch and Hegerfeldt [2]. Aharonov *et al* [1] show that the quantum radiation field possesses 'classical characteristics' only if it is in Glauber's coherent state [3]. By classical characteristics they [1] mean the attribute of 'indistinguishability of the radiation in two separate channels, whether it has been produced by independent sources or by a single source whose output is divided between the channels'. We will discuss this result in section 3. There it will be shown that this is a special case of bifactorizability. Thus, the single channels ('source') case involves, in fact, a tacit, factorized term which describes the vacuum of the other transformed coordinate. We also discuss in section 3 the results of Emch and Hegerfeldt [2] who show 'that if two quantum systems are prepared independently, and if their centre of mass is found to be in a coherent state, then each of the component systems is also in a coherent state'. This again is a special case of our considerations. In fact, we prove the following: if two quantum systems are prepared independently, and if their centre of mass is found to be in a pure state, then each of the component systems is also in a pure state, which in the coordinate representation is a Gaussian wavefunction.

In section 2 we define our problem and present our main mathematical results: basically we study bifactorizable (BF) quantum wavefunctions, i.e. cases when the wavefunction of a composite system can be written in two alternative ways (from now on, x will denote coordinates only):

$$\Psi(x_1, x_2) = \Psi_1(x_1)\Psi_2(x_2) \quad (1)$$

and also as

$$\Psi(x_1, x_2) = \Psi_3(x'_1)\Psi_4(x'_2) \quad (2)$$

with

$$\begin{aligned} x'_1 &= x_1 \cos \theta + x_2 \sin \theta \\ x'_2 &= -x_1 \sin \theta + x_2 \cos \theta \end{aligned} \quad (3)$$

for some given angle θ . We shall show that this property of bifactorization leads to factorizability for any rotation (arbitrary θ in equation (3)) if it is BF for a particular θ . It then follows that each of the $\Psi_i(x_i)$ must be of Gaussian form which is closely related to the so-called coherent state when pure states are studied. A more general relation between x_i and x'_i is also considered, as well as bifactorizability for operator-valued functions and for quantum density matrices. The problem of characterizing the normal distribution has a long and distinguished history going back to Maxwell and Hershel [4]. The proof presented here which is a generalization of the AFPL [1] approach leads to a novel functional characterization of the Gaussian function (equation (11)).

2. Alternative factorizations—basic results

In this section we derive properties of BF functions and BF functions of operators. The validity of our proofs is claimed for physically acceptable functions that are assumed to have the property of normalizability and differentiability.

(a) Functions $\Psi_1, \Psi_2, \Psi_3, \Psi_4$, which satisfy the functional equation (the BF condition):

$$\Psi_1(x_1)\Psi_2(x_2) = \Psi_3(x'_1)\Psi_4(x'_2) = \Psi_3(\mu x_1 + \nu x_2)\Psi_4(-\nu x_1 + \mu x_2) \quad (4)$$

with, for some $\theta \neq 0, n\pi/2$

$$\mu = \cos \theta \quad \nu = \sin \theta \tag{5}$$

are Gaussian, and each side of equation (4) is of the form

$$\Psi_1(x_1)\Psi_2(x_2) \sim \exp(-\lambda x^2 + \mathbf{b} \cdot \mathbf{x}) \tag{6a}$$

$$\Psi_3(x'_1)\Psi_4(x'_2) \sim \exp(-\lambda x'^2 + \mathbf{b}' \cdot \mathbf{x}'). \tag{6b}$$

Here \mathbf{x}, \mathbf{x}' are two-dimensional vectors,

$$\mathbf{x} = (x_1, x_2) \quad \mathbf{x}' = (x'_1, x'_2) \tag{7}$$

λ is a complex number with $\text{Re } \lambda \geq 0$ (for normalizability) and \mathbf{b} is some complex two-dimensional vector:

$$b'_1 = b_1 \cos \theta + b_2 \sin \theta$$

$$b'_2 = -b_1 \sin \theta + b_2 \cos \theta.$$

This can be seen as follows. We note that the BF condition (equation (4)) implies that $\Psi_i(x)$ cannot vanish anywhere (except at $|x| \rightarrow \infty$) because if $\Psi_i(x_0) = 0$ ($i = 1$ or $i = 2$) for some x_0 then Ψ_3 and/or Ψ_4 must vanish identically. We thus normalize

$$\Psi_i(0) = 1 \quad i = 1, 2, 3, 4. \tag{8}$$

Now for $x_1 = 0$, from equation (4) using equation (8) it follows that

$$\Psi_2(x_2) = \Psi_3(\nu x_2)\Psi_4(\mu x_2). \tag{9a}$$

Similar reasoning with $x_2 = 0$ yields

$$\Psi_1(x_1) = \Psi_3(\mu x_1)\Psi_4(-\nu x_1). \tag{9b}$$

By substituting equation (9) in equation (4), renaming the variables $\mu x_1 \rightarrow x, \nu x_2 \rightarrow y$ and calling $\gamma = \cot \theta \equiv \mu/\nu$, we obtain

$$\frac{\Psi_3(x+y)}{\Psi_3(x)\Psi_3(y)} = \frac{\Psi_4(-x/\gamma)\Psi_4(\gamma y)}{\Psi_4(-x/\gamma + \gamma y)} \equiv F(x, y). \tag{10}$$

This functional equation can be analysed by expanding both sides in a Taylor series about the origin and, upon equating the coefficients of equal powers of x and y one is led to the result given in equation (6). Rather than pursuing the above 'brute force' path we demonstrate this result (equation (6)) by the following argument. From the left-hand side of equation (10), it is clear that $F(x, y)$ is symmetric with respect to the interchange $x \leftrightarrow y$. Hence it is of the general form of $F[(x+y), xy]$. However, the term involving γ (mid-term of equation (10)) implies that $F(x, y)$ cannot depend on $x+y$, i.e. we have

$$F(x, y) \equiv f(xy) = \frac{\Psi(x+y)}{\Psi(x)\Psi(y)} \tag{11}$$

where the subscript 4 was dropped for simplicity. We now consider this identity near the point $(x; y = 0)$ and choose $\delta y = \delta x$. Using equation (8) (and hence also $f(0) = 1$ (cf equation (11))) and denoting differentiation by a prime we get to first order in δx :

$$f(x\delta x) = 1 + xf'(0)\delta x$$

$$\Psi(x + \delta x) = \Psi(x) + \Psi'(x)\delta x$$

$$\Psi(\delta x) = 1 + \Psi'(0)\delta x.$$

Upon substituting in equation (11) we get $\Psi'(x)/\Psi(x) = \Psi'(0) + xf'(0)$; hence

$$\Psi(x) \sim \exp(\frac{1}{2}f'(0)x^2 + \Psi'(0)x) \tag{12}$$

which is indeed Gaussian. The same holds for all the functions Ψ_i ($i = 1-4$), and the same λ (i.e. $\frac{1}{2}f'(0)$ of equation (12)) is common to all the functions Ψ_i . It should be noted that the general functional equation defining a Gaussian is that given by equation (11). This is a generalization of the functional equation defining the exponential function: $E(x)E(y) = E(x + y)$, implying $E(x) = \exp(\alpha x)$.

For general complex λ , each Ψ_i represents a squeezed state. If we require that the first-order quantum correlation [5] vanishes, i.e.

$$\frac{1}{2}\langle pq + qp \rangle - \langle p \rangle \langle q \rangle = 0$$

then it follows that $\text{Im } \lambda = 0$ [5]. In this case the function

$$\Psi(x) \sim \exp(-\lambda x^2 + bx)$$

(λ real) is an eigenfunction of the boson annihilation operator

$$a = \sqrt{\lambda}x + \frac{ip}{2\hbar\sqrt{\lambda}}.$$

It then follows that $\Psi(x)$ is a Glauber coherent state for the operator a .

(b) The Gaussian property of BF functions also holds for a general linear transformation, i.e. if

$$\Psi_1(x_1)\Psi_2(x_2) = \Psi_3(x'_1)\Psi_4(x'_2) \tag{13}$$

where

$$\begin{aligned} x'_1 &= T_{11}x_1 + T_{12}x_2 \\ x'_2 &= T_{21}x_1 + T_{22}x_2 \end{aligned} \tag{14}$$

(with $T_{11}T_{12}T_{21}T_{22} (T_{11}T_{22} - T_{12}T_{21}) \neq 0$) then each Ψ_i is a Gaussian (because by a rescaling of all the variables, equation (14) will assume a form similar to the one given by equation (3)).

(c) Consider a BF function of boson creation and annihilation operators,

$$f_B(a_B^\dagger)f_C(a_C^\dagger) = f_A(a_A^\dagger)f_D(a_D^\dagger) \tag{15}$$

with the different f_i ($i = A, B, C, D$) characterizing different fields. However, the fields are related by

$$a_A^\dagger = \mu a_B^\dagger + \nu a_C^\dagger \tag{16}$$

$$a_D^\dagger = -\nu^* a_B^\dagger + \mu^* a_C^\dagger$$

$$|\mu|^2 + |\nu|^2 = 1 \tag{17}$$

i.e. here the operators are connected via an SU(2) transformation. Here again we use the normalization

$$f_i(0) = 1. \tag{18}$$

This case may be analysed with complex quantities z_i instead of the real quantities x and y (e.g. by taking the matrix element of equation (15) between the ground state and the coherent state $\langle z_i^\dagger 1$; see [1]). Then the functional form will again emerge to

be Gaussian except for phase factors that are related to the transformations defined by equations (6) and (7). The SU(2) transformation is defined by the complex quantities μ and ν . Two separate phases may be given by

$$\frac{\mu^*}{\mu} = e^{2i\varphi_\mu} \quad \text{and} \quad \frac{\nu^*}{\nu} = e^{2i\varphi_\nu}$$

then the four functions may be given as

$$\begin{aligned} f_A(z) &\sim \exp\left(\frac{1}{2}G'(0) e^{2i(\varphi_\mu + \varphi_\nu)} z^2\right) \\ f_B(z) &\sim \exp\left(\frac{1}{2}G'(0) e^{2i\varphi_\nu} z^2\right) \\ f_C(z) &\sim \exp\left(\frac{1}{2}G'(0) e^{2i\varphi_\mu} z^2\right) \\ f_D(z) &\sim \exp\left(\frac{1}{2}G'(0) z^2\right) \end{aligned} \tag{19}$$

with

$$G(z, z') = \frac{f_D(z+z')}{f_D(z)f_D(z')}$$

Here we have left out the term linear in z in the exponent to emphasize the Gaussian nature of the individual functions. The universality of the Gaussian functional form is true except for a phase factor that is important in the context of the SU(2) transformation. We note that the action of $f_i(a_i^\dagger)$ on the vacuum generates a squeezed state, unless $G'(0) = 0$ in which case it reduces to a coherent state.

Finally, we note that the Gaussian property of the BF wavefunctions is carried over into BF density matrices: suppose that a density matrix of a two-particle system is BF, i.e.

$$\langle y_1 y_2 | \rho | x_1 x_2 \rangle = \langle y_1 | \rho_1 | x_1 \rangle \langle y_2 | \rho_2 | x_2 \rangle = \langle y'_1 | \rho_3 | x'_1 \rangle \langle y'_2 | \rho_4 | x'_2 \rangle. \tag{20}$$

(Here again the primed coordinates are related to the unprimed, for each particle, by an O(2) transformation, equation (3).) Then it follows that

$$\langle y_i | \rho_i | x_i \rangle = \exp(-ax_i^2 - by_i^2 + cx_i y_i + d_i x_i + e_i y_i) \quad i = 1, 2 \tag{21}$$

with a similar equation for the primed variables. The constants a , b and c are common to all four density matrices. (The case of $c = 0$ in the above equation corresponds to a pure-state density matrix.) To prove this assertion we merely note that holding fixed one set of coordinates (e.g. the y -coordinates), we are back to the case of ordinary functions satisfying the BF condition, and therefore each ρ_i must be Gaussian in its x_i with coefficients which depend on y_i . Since the same argument holds when we fix the x -coordinates, it follows that the only terms which may be added to the bilinear form in the exponent in equation (21) are of the form $x_i y_i^2$, $x_i^2 y_i^2$, $x_i^2 y_i$. However, it is easily verified that the inclusion of such terms will violate equation (20), unless their coefficients vanish.

3. Factorizations of special interest

In this section we study two bifactorizations which are of special interest. The first involves two independent particles whose wavefunction is a product of functions of the individual coordinates and, further, the wavefunction is also a product when expressed in the centre of mass and relative coordinates, i.e.

$$\Psi'_1(x_1)\Psi'_2(x_2) = \Psi'_3(\mu_1 x_1 + \mu_2 x_2)\Psi'_4(x_2 - x_1). \tag{22}$$

Here $\mu_i = m_i/(m_1 + m_2)$, with m_i the mass of the i th particle, and x_i its coordinate. It is easy to see that by a rescaling of the variables, equation (22) can be brought to the form

$$\Psi_1(x'_1)\Psi_2(x'_2) = \Psi_3(x''_3)\Psi_4(x''_4) \quad (23)$$

where

$$x''_3 = x'_1 \cos \theta + x'_2 \sin \theta \quad (24)$$

$$x''_4 = -x'_1 \sin \theta + x'_2 \cos \theta$$

$$x'_i = \sqrt{\mu_i} x_i \quad i = 1, 2 \quad (25)$$

$$\cos \theta = \sqrt{\mu_1} \quad \sin \theta = \sqrt{\mu_2} \quad (26)$$

and

$$\begin{aligned} \Psi_1(x) &= \Psi'_1\left(\frac{x}{\sqrt{\mu_1}}\right) & \Psi_2(x) &= \Psi'_2\left(\frac{x}{\sqrt{\mu_2}}\right) \\ \Psi_3(x) &= \Psi'_3(x) & \Psi_4(x) &= \Psi'_4\left(\frac{x}{\sqrt{\mu_1\mu_2}}\right). \end{aligned} \quad (27)$$

It then follows that the general form of Ψ_i (in the rescaled variables) is of the form of equation (6). Some of the results of [2] can be considered as a special case of the above considerations. Indeed, let there be given a density matrix of two independent particles. Then it may be written as a product of the density matrices of the individual particles. If one assumes that the centre of mass of the system is in a pure state, then a theorem by Von Neumann [6] assures us that the two-particle density matrix factorizes into a product of a pure-state density matrix in the centre of mass and a density matrix for the relative coordinate. Hence, we have bifactorizability of the density matrix, and this, together with the information that one of the density matrices involved (the CM) is in a pure state, implies that all the density matrices involved are pure-state density matrices and, moreover, are of the general Gaussian form; under the added constraint that the wavefunction of the centre of mass is a coherent state, then λ is fixed to be real and we get that all Ψ_i are now coherent states, which is the result obtained by Emch and Hegerfeldt [2] in a different way.

The second bifactorization of interest is the one implicit in [1]. Comparing their treatment with ours (see equation (15)), it is clear that the D mode is not excited in their case, i.e. they chose $f_D \equiv 1$. Therefore $G \equiv 1$, implying $G'(0) = 0$, and hence f_A , f_B and f_C , in equation (19), reduce simply to exponential functions whose exponent is linear in z (see remark below equation (19)), thus $f_i(a_i^+) \propto \exp(\alpha_i a_i^+)$, i.e. coherent states. Alternately, if written in the 'coordinate' representation, AFPL treatment is equivalent to assuming, in our equation (12), that one of the functions is $e^{-\lambda x^2}$ with $\lambda > 0$ and $\Psi'(0) = 0$ (this is what is meant here by one channel empty). As our treatment showed that λ is common to all functions we have that all the Ψ_i involved are coherent states.

4. Conclusions

Bifactorizable wavefunctions were defined as functions of two variables which can be factorized in two independent ways. These functions were shown to satisfy a functional

equation which is a generalization of the functional equation for the exponential function. The general form of these functions was shown to be Gaussian and for rotations the functions could then be factorized in infinite ways. We related the results to known [1, 2] theorems on the properties of coherent states and, in part, generalized them.

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